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THE ACTION OF A RIGID STAMP ON A HALF-PLANE WEAKENED BY A REGULAR SYSTEM OF CRACKS*

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The action of a rigid stamp on an isotropic half-plane weakened by a regular system of rectilinear cracks is studied. The problem in question differs fundamentally from the wide class of problems solved in recent times for symmetric bodies in the non-periodic character of the boundary conditions at the edge of the half-plane.

The unknown function of the contact pressures is sought in the form of an expansion in terms of Chebyshev polynomials. The coefficients of this expansion are found from the system of algebraic equations obtained by transforming the condition of compatibility of the vertical displacements of the stamp and its foundation. A series of problems arising in this connection and concerning the stress-strain state of the half-plane under the action of loads applied to its edge described in terms of Chebyshev polynomials, is solved using a general scheme which makes use of the symmetric properties of the medium. Results of a numerical analysis and the functions of contact pressures under the stamp and the stress intensity coefficients at the crack tips are given.

A smooth stamp is situated on the segment $\gamma = [d-h, d+h]$ of the x axis. The function $f(x)$ describes the shape of its base. The forces acting on it are reduced to the principal vector Q and moment M. We assume that the area of contact between the stamp and foundation does not change, and that there is no interaction between the crack edges. Fig.1 shows the orientation of the cracks, the position of the coordinate axes and the basic geometrical dimensions, and gives the notation.

Although the system of cracks is regular, the medium in question is not geometrically symmetrical. The symmetry is violated by the mixed boundary conditions at the edge of the half-plane. As far as the authors know, a problem of this class has not been solved before.

The proposed scheme of investigation can be extended to a large number of other classical contact problems for foundations with a periodic structure.

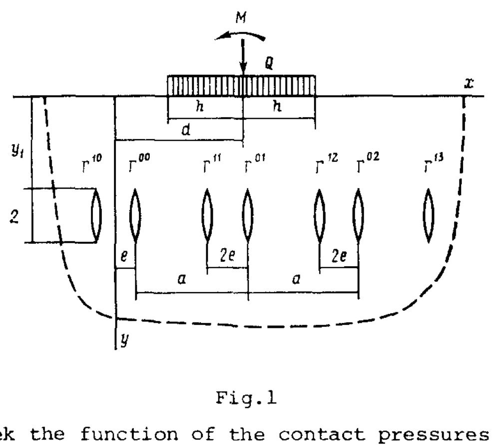

Fig.1

We shall naturally seek the function of the contact pressures $p(x)$ under the stamp in the form

$$p(x) = \sum_{k=0}^{\infty} a_k p_k(x), \quad p(x) = \frac{T_k(\xi)}{\sqrt{1-\xi^2}} \tag{1}$$

$$\xi = \omega(x) = (x-d)/h, \quad \xi \in [-1, 1]$$

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where $T_k(\xi)$ are Chebyshev polynomials of first kind and a_k $(k = 0, 1, \ldots)$ are real coefficients to be determined.

In accordance with expansion (1) and the principle of superposition, we have

$$v(x) = \sum_{k=0}^{\infty} a_k v_k(x), \quad x \in \gamma \tag{2}$$

$$\chi^{(js)(p)}(y) = \sum_{k=0}^{\infty} a_k \chi_k^{(js)(p)}(y), \quad y \in \Gamma^{js} \tag{3}$$

$$(p = 1, 2;\ j = 0, 1;\ s = 0, \pm 1, \pm 2, \ldots)$$

Here $v(x)$ and $\chi^{(js)(p)}(y)$ are the vertical displacements of the points of the segment γ, and the jumps in the partial derivatives in y of the horizontal displacement $(p = 1)$ and vertical displacements $(p = 2)$ of the points of the half-plane during passage through the segment Γ^{js}, while $v_k(x)$ and $\chi_k^{(js)(p)}(y)$ are their components corresponding to the load $p_k(x)$. Constructing the jumps $\chi^{(js)(p)}(y)$ solves the problem in question completely.

Let us first pause and consider the determination of the functions $\chi_k^{(js)(p)}(y)$. The corresponding load $p_k(x)$ is applied to the segment γ. Naturally, we decompose the stress-strain state into the basic state occurring in a continuous half-plane, and the perturbed state which compensates for the stresses occurring at the crack edges in the basic state. Since in the latter state we have no jumps during the passage through the segments Γ^{js}, it follows that we can find the functions which are of interest by solving just a single perturbed state. Here the edge of the half-plane will be load-free and the following normal and shear loads will be applied to the crack edges Γ^{js}:

$$\sigma_x^{(k)js}(y) = -\int_{d-h}^{d+h} \sigma_x^F(e_{js} - x, y)\, p_k(x)\, dx \tag{4}$$

$$\tau_{xy}^{(k)js}(y) = -\int_{d-h}^{d+h} \tau_{xy}^F(e_{js} - x, y)\, p_k(x)\, dx$$

$$(e_{js} = (-1)^j e + sa;\ j = 0, 1;\ s = 0, \pm 1, \pm 2, \ldots)$$

where $\sigma_x^F(x, y)$ and $\tau_{xy}^F(x, y)$ denote the corresponding stresses in the well-known Flamant problem for a continuous half-plane /1/.

The convergent method for studying the perturbed state is that given in /2/, based on the already approved scheme of solving problems of linear mechanics for symmetric bodies /3/. Therefore, we shall regard, from now on, the jumps $\chi_k^{(js)(p)}(y)$ as known. In this case we can easily find any components of the stress-strain state in question under the load $p_k(x)$. In particular, we have

$$v_k'(x) = v_k^{(0)\prime}(x) + v_k^{(1)\prime}(x), \quad x \in \gamma \quad (v' = dv/dx) \tag{5}$$

$$v_k^{(0)\prime}(x) = \frac{2}{\pi E} \int_{d-h}^{d+h} \frac{T_k(\xi)\, ds}{\sqrt{1-\xi^2}\,(s - x)} \tag{6}$$

$$v_k^{(1)\prime}(x) = \frac{1}{\pi E} \sum_{p=1}^{2} \sum_{j=0}^{1} \sum_{s=-\infty}^{\infty} \int_{y_1}^{y_1+2} \chi_k^{(js)(p)}(y)\, K^{(js)(p)}(x, y)\, dy \tag{7}$$

$$(k = 0, 1, 2, \ldots)$$

$$K^{(js)(1)}(x, y) = 2(e_{js} - x)\, y^2/\Delta_{js}^2$$

$$K^{(js)(2)}(x, y) = y/\Delta_{js} + y\,[y^2 - (e_{js} - x)^2]/\Delta_{js}^2$$

$$\Delta_{js} = (e_{js} - x)^2 + y^2$$

where the superscripts 0 or 1 denote quantities connected with the basic or the perturbed state of the half-plane, and E is the modulus of elasticity.

We will now turn our attention to the problem of calculating the coefficients a_k $(k = 0, 1, 2, \ldots)$ in expansions (1)-(3). The usual conditions of equilibrium of the stamp yield at once

$$a_0 = Q/(\pi h), \quad a_1 = 2M/(\pi h) \tag{8}$$

In order to find the remaining coefficients, we shall construct a linear system of algebraic equations. According to the contact conditions of

$$v(x) = V + (x - d)\,\varphi + f(x), \quad \forall x \in \gamma \tag{9}$$

where V and φ are the displacement and angle of rotation of the stamp. Equating the right-hand sides of relations (2) and (9), differentiating the result and taking formulas (8) into account, we find

$$\sum_{k=2}^{\infty} a_k v_k'(x) = \varphi + F(x), \quad x \in \gamma \tag{10}$$

$$F(x) = f'(x) - \frac{Q}{\pi h} v_0'(x) - \frac{2M}{\pi h} v_1'(x) \tag{11}$$

Let us multiply expression (10) by $U_m(\xi)\sqrt{1-\xi^2}$, where $U_m(\xi)$ is a Chebyshev polynomial of the second kind, and integrate it on the segment γ. Repeating this operation for $m = 0, 1, \ldots$ and taking formula (5) and the relation $v_k^{(0)'}(x) = 2U_{k-1}(\xi)/E$ $(k = 1, 2, \ldots)$, following from (6) into account, we can construct the following system for determining the coefficients:

$$\sum_{k=2}^{\infty} A_{1k} a_k = \frac{\pi}{2}\varphi + B_1 \tag{12}$$

$$a_m + \frac{E}{\pi}\sum_{k=2}^{\infty} A_{mk} a_k = \frac{E}{\pi} B_m \quad (m = 2, 3, \ldots) \tag{13}$$

The quantities A_{mk} and B_m are found from the relations

$$A_{mk} = \int_{d-h}^{d+h} v_k^{(1)'}(x) U_{m-1}(\xi)\sqrt{1-\xi^2}\,dx \tag{14}$$

$$B_m = \int_{d-h}^{d+h} F(x) U_{m-1}(\xi)\sqrt{1-\xi^2}\,dx \quad (m = 1, 2, \ldots; k = 2, 3, \ldots)$$

Without pausing to consider the technical aspects of numerical integration, we shall show that after solving the system of Eqs.(13) and determining the coefficients a_k $(k = 2, 3, \ldots)$, we can substitute them, together with (8), into (12), (1) and (3), and this will yield the rotation of the stamp, the function of the contact pressures and the jumps in the derivatives caused by the displacements at the cracks.

Let us give some numerical results for a purely periodic system of cracks $(e = a/4)$ and a flat stamp $f(x) = 0$, acted upon by a central force Q $(M = 0)$.

Table 1 gives data on the coefficients K_n of the intensity of normal stresses at the upper $(n = 1)$ and lower $(n = 2)$ tips of the cracks Γ^{00} $(N = 1)$, Γ^{11} $(N = 2)$, Γ^{01} $(N = 3)$ at $a = 4$, and various relations between other parameters and a fixed value $Q/h = 1$, and the values quoted are for $10^3 \cdot K_n$. It is best to show that the coefficients K_n are determined from the jumps $\chi^{(j,l)(1)}(y)$ at the corresponding cracks, using the known relations /4/, and in the case of a single crack situated in an isotropic plane and acted upon by internal unit pressure they are equal to unity.

Table 1

y_1	h	$N = 1$		2		3	
		$n = 1$	2	1	2	1	2
0.125	1	161	73	29	41	2	13
	0.25	40	21	5	9	0	4
	0.5	81	40	12	18	0	7
0.25	1	119	60	37	44	5	16
	2	248	112	102	102	8	34
	3	264	180	168	93	54	78
0.5	3	231	153	126	81	78	87

In what follows, we shall assume that $a = 4$, $y_1 = 0.25$ and $d = 0$.

Fig.2 shows the functions of the contact pressures under the stamps of various lengths (the dashed lines correspond to the continuous half-plane). We see that for short stamps situated between the cracks the presence of the latter leads to an increase in the mean values of the contact pressures, and to their reduction at the peripheral zones of the base. In the case of long stamps, the presence of the cracks in the half-plane leads to a characteristic oscillation connected with the decrease in contact pressures above the cracks and their increase between them.

If we assume that the cracks were preloaded with a uniform internal pressure of intensity q, then the values Q^* and Q^*/h at which the cracks begin to close will be of interest from the point of view of engineering applications in which the crack edges may come into contact with each other. It is obvious that for $Q \leqslant Q^*$ the above solution remains valid. Numerical studies have shown that the first cracks to close are Γ^{00} and Γ^{10}.

The relations connecting Q^*/q and $Q^*/(hq)$ with h are shown by lines *1* and *2*, respectively in Fig.3. The dashed lines show their asymptotes, which have an obvious physical meaning. The ordinate of point *A* of line *1* corresponds to a stamp of zero length, and is identical with the value Q^* of the analogous problem formulated for the half-plane in question, loaded by a force applied to the origin of coordinates. The bumps in the lines *1* and *2* near the value $h = 1$ are explained by the fact that the contact pressures above the crack have, according to the solution of the Flamant problem, little effect on its expansion.

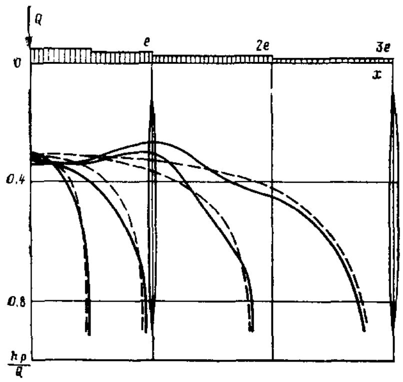

Fig.2

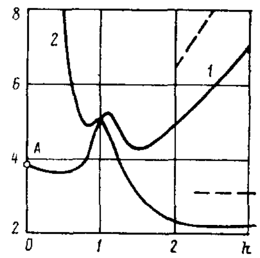

Fig.3

In conclusion we note, that the reduction method was used to solve system (13). Its level depends essentially on the magnitude of the parameter h/a.

In the case of short stamps, i.e. for small values of h/a, the sums (1)-(3) converge very rapidly and this makes it possible to retain in them a small number (k_0) of terms to achieve acceptable accuracy. This assertion is illustrated by the data given in Table 2, which gives the values of $10^3 \cdot p(x)$, where $p(x)$ denote the contact pressures at the points $x_j = jh/5$ $(j = 0, 1, \ldots, 4)$. They were computed for various values of k_0 for $e = 1, h = 1$.

Table 2

k_0	$j = 0$	1	2	3	4
4	355	360	375	411	513
6	333	344	377	434	544
10	336	344	374	436	546
14	335	344	375	435	547

In the case of longer stamps, the number of coefficients a_k $(k = 2, 3, \ldots, k_0 - 1)$, obtained from system (13) should be increased in proportion to h/a.

REFERENCES

1. REKACH V.G., Manual for Solving Problems in the Theory of Elasticity. Vyssh. Shk., Moscow, 1966.
2. BURYSHKIN M.L. and RADIOLLO M.V., Generalized periodic problem of the theory of elasticity for a half-plane with cracks. PMM, 49, 2, 1985.
3. BURYSHKIN M.L., A general scheme for solving inhomogeneous linear problems for symmetrical mechanical systems. PMM, 45, 5, 1981.
4. PANASYUK V.V., SAVRUK M.P. and DAPYSHIN A.P., The Stress Distribution Around Cracks in Plates and Shells. Nauk. Dumka, Kiev, 1976.

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